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Fast, Adaptive Finite Element Scheme for Viscous Incompressible Flows

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This paper presents an adaptive finite element procedure for solving viscous incompressible flows. The methodology is based on adaptive remeshing for the solution of the steady-state Navier-Stokes equations for an incompressible fluid. Solutions are obtained in primitive variables by an Uzawa algorithm using a highly accurate element. Two error estimators are presented and compared for a problem with a known analytical solution. The methodology is then applied to a problem of practical interest, and predictions are compared with experimental measurements. The proposed adaptive scheme is shown to lead to improved accuracy of the finite element predictions.

Nomenclature

C	= constant
E	= velocity error
e	= error in the solution
h	= element size
$J(\,)$	= dissipation energy
L	= plate length
n	= number of elements in the mesh
p	= pressure
Q	= pressure error
q	= test function for pressure error
S	= step and fence height
U	= velocity vector
u, v	= velocity components
V	= test function for velocity error
x, y	= coordinates
δ	= element size predicted by adaptation module
ϵ	= strain rate tension, $\frac{1}{2}(\nabla U + \nabla U^T)$
η	= relative error on a mesh
ξ	= similarity variable in a boundary layer
μ	= absolute viscosity of the fluid
ν	= kinematic viscosity of the fluid
σ	= deviatoric stress sensor, $2\mu\epsilon$
∇	= gradient operator

Introduction

OVER the past few years, adaptive methods have stirred much interest because of their ability to provide accurate solutions by automatically clustering grid points near flow features of interest such as shock waves and boundary layers. The first inroads were achieved in compressible flow applications because of the pressing need for proper prediction and computation of shock waves. See Refs. 1–3 for a review of recent work.

Adaptive methods offer the means of tackling complex flow problems at a reasonable cost. Indeed, such methods can compute the most accurate solution possible for fixed computer resources, or they can achieve a preset level of accuracy with a minimum amount of CPU time. Adaptive methods thus provide a framework for controlling the quality and reliability of numerical simulations.

Whereas spectacular results were achieved for both two- and three-dimensional aerodynamic flow problems,[4] very little has been accomplished for viscous incompressible flows. Proof of concept computations were recently reported for viscous incompressible flows.[6–8] This paper focuses on improving the reliability, robustness, and computational efficiency of the methodology proposed by the authors in Ref. 8. Improvements are achieved by replacing the postprocessing error estimator by a mathematically more rigorous one based on the solution of partial differential equations for both the velocity and the pressure errors.

The method is based on adaptive remeshing coupled to a finite element solver for viscous incompressible flows. The proposed adaptive methodology is designed to transform an existing finite element flow solver into an adaptive one. This modular approach makes it possible to replace the flow solver if and when a more efficient one is developed. Adaptivity is thus completely separate from the flow solver. The adaptation module requires knowledge of only the finite element scheme used to discretize the equations. It does not need to know how the nonlinear algebraic equations are solved. The work of the Swansea group[3,4] involved adaptive remeshing for compressible flows, whereas that of Löhner used h refinement for the Euler equations. In both cases, error estimates are derived from interpolation theory applied to a key variable of the flow, usually density. The present work concentrates on the rigorous development and validation of error estimators specifically designed for the Navier-Stokes equations for an incompressible fluid. These error estimations are used to generate an improved mesh by the advancing front technique.

The paper is organized as follows: The section on methodology discusses issues specific to adaptivity for incompressible flows. Mesh generation requirements, finite element algorithm, error estimation, and adaptive strategy are presented. The proposed methodology is then validated by applying it to a problem with a known analytical solution for which improvement in solution quality can be easily quantified. Although it is of academic interest, this test problem presents features of flows of practical interest, such as thin boundary layers. Finally, the method is applied to a flow for which experimental data are available.

General Methodology

There are several ways of achieving adaptivity:

1) The degree of polynomial approximation can be changed locally. These are the so-called p methods.[13]

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2) The grid points can be relocated. This is called an r method.

3) The grid can be refined locally by either element subdivision or by regenerating a new mesh to achieve higher local grid density for improved accuracy. These approaches are called h refinement and remeshing methods, respectively.

Adaptive remeshing has been selected because it provides the greatest control of element size, stretching, and orientation to better resolve flow features such as boundary layers or stagnation and reattachment points. Adaptive remeshing also offers the possibility of transforming an existing incompressible finite element flow code into an adaptive solver.

In remeshing, the problem is first solved on a grid fine enough to roughly capture the physics of the flow. The resulting solution is then analyzed to determine where more grid points are needed, and an improved mesh is generated. The process is repeated until the required level of accuracy is achieved.

Incompressibility Considerations

Remeshing offers an elegant yet simple approach for overcoming some of the obstacles specific to incompressible viscous fluid flows. The best proven finite element approximations can be selected based on their convergence and accuracy properties.[8,9] Such an element is the so-called seven-noded Crouzeix-Raviart triangle.

The use of such elements imposes some specific requirements on the part of mesh generators. Most aerodynamics triangulation codes produce three-noded triangles that are unsuitable for the present application. Hence, following Ref. 3, a mesh generator was developed that supports higher-order elements. It turns out that, for adaptive remeshing, such higher-order elements have the added benefit of being very cost effective. Their higher-order convergence rate is not only of theoretical importance but also means that fewer degrees of freedom will be required to achieve a preset level of accuracy. In practice this often makes the difference between being able to reach a given accuracy and failing to reach it.[8]

The Navier-Stokes equations are solved in primitive variables using the Galerkin finite element method. The seven-node Crouzeix-Raviart element is used to discretize the equations of motion.[8] The finite element equations are assembled into a compacted skyline format. The global system of equations are linearized by Newton's method and solved using Gaussian elimination. An augmented Lagrangian (Uzawa) formulation is used to treat incompressibility. In this approach, pressure degrees of freedom are eliminated from the global system.[9] Thus, significant computational savings are achieved.

Error Estimation

Error estimation is central to the success of adaptive methods. A poor error estimate can result in too many elements being generated in regions of smooth flow or in too coarse a grid in regions where the solution changes rapidly. A good estimator should have the following characteristics:

1) It must be rooted in the physics of the problem.

2) It must have a mathematical foundation.

3) It must be convergent; i.e., as the number of degrees of freedom is increased, the estimator should approach the true error.

4) It must not be too expensive to compute.

It should be mentioned that, although an error estimator should be sensitive to singularities, it should not be so overly sensitive that regions with sharp but smooth gradients are overlooked.

Two error estimators have been retained for this study. The first one is based on postprocessing of the stress and strain rate tensors, and the second one entails the solution of partial differential equations for the velocity and pressure errors. This estimator is mathematically more rigorous for the Navier-Stokes equations and results in more robust and reliable error estimates. The partial differential equations for the errors are solved on an element-by-element basis so that the cost of estimating the error represents only a small fraction of the cost of solving the flow problem itself.

Postprocessing Error Estimation

A relatively simple error estimator based on postprocessing was presented in Refs. 8 and 11. For incompressible creeping flow problems, such an estimator can be derived from the variational formulation of Stokes flow. It is necessary to find a velocity $\boldsymbol{U}$ with zero divergence that minimizes the dissipation energy functional:

$$J(\boldsymbol{U}) = \int \frac{\mu}{2} (U_{i,j} + U_{j,i})^2 \, \mathrm{d}V = \int \sigma{:}\epsilon \, \mathrm{d}V \tag{1}$$

If $\boldsymbol{U}_h$ is an approximate finite element solution and $\boldsymbol{U}_{\text{ex}}$ is the exact solution, the dissipation energy of the error (energy of the error),

$$e = \boldsymbol{U}_{\text{ex}} - \boldsymbol{U}_h$$

can be evaluated by computing $J(e)$:

$$J(e) = \int (\sigma_h - \sigma_{\text{ex}}) : (\epsilon_h - \epsilon_{\text{ex}}) \, \mathrm{d}V \tag{2}$$

Obviously, the exact solution will have zero error energy. As can be seen, this estimator satisfies the first two criteria cited previously: It is rooted in the physics and has a rigorous mathematical foundation for creeping flow problems.

This estimator will not have as strong a foundation for the Navier-Stokes equations as it has for Stokes flow because there exists no functional whose minimization yields a solution to the Navier-Stokes equations. Nevertheless, it should still prove of some use. This estimator is sensitive to high strain rates and shear stresses by construction. Thus, it should react favorably to boundary layers, stagnation points, jets and wakes.

Numerical simulations do confirm this property but also indicate one major shortcoming of this estimator; it is not sensitive to pressure variations. Numerical simulations will show that reattachment points are not well recognized by this estimator because stresses and strains are small compared to pressure variations.

Unfortunately, the exact solution required by Eq. (2) is not available in cases of practical interest. Hence, an approximation to $J(e)$ must be computed. This can be accomplished by noting that whereas real stresses and strains are continuous, the finite element scheme produces stresses and strains that are discontinuous across element interfaces. It is a simple matter to recover continuous stresses and strains from the finite element solution through a least-squares projection.[6] The difference between the continuous and discontinuous field can then be used as an indication of the accuracy of the solution. This approach can be formally proven for linear problems because of the superconvergence properties of the finite element method.[12] The estimator, using smoothed stresses σ_s and strains ϵ_s now takes on the following form:

$$J(e) = \int (\sigma_h - \sigma_s) : (\epsilon_h - \epsilon_s) \, \mathrm{d}V \tag{3}$$

The error estimation for a given element is obtained by integrating Eq. (3) over the area of the element of interest. The global error is computed by integration over the whole domain. Finally, the energy norm of the error is obtained through

$$||e_{\text{tot}}||^2 = J(e)$$

Local Partial Differential Equation Problem for the Error

Following Ref. 13, a set of partial differential equations for the velocity and pressure errors has been derived from the

Navier-Stokes equations. They are solved on each element in turn. This results in the following variational problem for the velocity and pressure errors:

$$\int_e \mu(\nabla E + E^T) : \nabla V - Q \nabla \cdot V$$
$$= \int_e \{ \nabla \cdot \mu(\nabla u_h + \nabla u_h^T) - \nabla p - \rho u_h \cdot \nabla u_h \} V \qquad (4)$$
$$+ \frac{1}{2} \int_{\partial e} (t - t^*) V$$

$$\int_e q \nabla \cdot E = \int_e - q \nabla \cdot U_h \qquad (5)$$

where V and q are the test functions corresponding to E and Q, and $(t - t^*)$ represents the jump in stresses across element faces. The first term on the right-hand side of Eq. (4) is the element residual; it is a measure of how accurate the finite element solution is inside an element. The stress jump term reflects how smooth the solution is across an element side and is a local measure of how well the solution is approximated by two adjacent elements. Equation (5) is a measure of mass conservation.

This problem is solved locally on each element and results in small 9×9 systems of equations that are inexpensive to solve. Note the presence of both the pressure and convective terms on the right-hand side of Eq. (4). This should make the local problem sensitive to both velocity and pressure gradients.

The norm of the combined velocity and pressure error is computed using

$$||e||^2 = |||(E, Q)|||^2 = 2||E||^2 + \frac{1}{\mu}||Q||_0^2 \qquad (6)$$

where $||E||$ is computed using Eq. (2) and $||Q||_0$ is the L_2 norm of the presure error.

Adaptive Strategy

The next issue lies in how to adapt the mesh given one of the error estimates described previously. The adaptive remeshing strategy proceeds as follows:

1) Generate an initial mesh.
2) Compute the finite element solution.
3) Compute an error estimate.
4) if (relative error < tolerance)
   then
      stop
   else
      generate a new mesh according to error estimates,
      interpolate solution on new mesh and go to 2
   end if

As can be seen, the bulk of the work consists in designing an improved mesh from the error estimate. Once a finite element solution has been obtained, the error on each element of the mesh is evaluated using Eq. (2) or Eq. (6). The total error and the norm of the solution are computed according to:

$$||e_{\text{tot}}||^2 = \sum ||e_k||^2$$

so that the global relative error can be estimated:

$$\eta = ||e_{\text{tot}}|| / ||U||$$

where η is a measure of the global accuracy of the finite element solution. There remains to compute the element size for the improved mesh such that elements are smaller in regions of large error and bigger in regions where the solution is already accurate. To achieve this, one must invoke the concept of an optimal mesh. A mesh is said to be optimal if all finite elements of the grid have the same error. Thus, on such a mesh, the solution is uniformly accurate throughout the domain. Following Ref. 11, given a target relative error η_t, the total error can then be related to the yet unknown average element error:

$$n||e_{\text{av}}||^2 = \eta_t^2 ||U||^2$$

thus

$$||e_{\text{av}}|| = \eta_t ||U|| / n^{1/2}$$

where n is the number of elements in the mesh.

Finally, element sizes for the next grid can be computed from the finite element asymptotic rate of convergence,[6] which relates the error on element j to some power k of the element size h:

$$||e_j|| = Ch^k$$

This relationship can also be written for the target average error:

$$||e_{av}|| = C\delta^k$$

Solving for the element size, we get

$$\delta = \left[\frac{\eta_t ||U||}{||e|| n^{1/2}} \right]^{1/k} h$$

This distribution of element sizes is used in the advancing front mesh generator in order to generate an improved grid. In the present work $k = 2$ for the Crouziex-Raviart element

Applications

Two-Dimensional Boundary Layer

The two error estimators are first applied to a simple two-dimensional boundary-layer flow for which an analytical solution is known. This serves as a basis for comparing the estimators and for showing the increased accuracy afforded by the element-by-element estimator.

The flowfield is given by

$$U = 1 - \exp(-\xi)$$
$$P = x$$
$$\xi = y[U_0/(\nu x)]^{1/2}$$

The similarity variable ξ produces a velocity field closely resembling that of a boundary layer over a flat plate. The sample streamline plot of Fig. 1 clearly shows the thickening of the boundary layer. The Reynolds number, based on the plate length L, is equal to 200. The computational domain stretches from $x/L = 0.1$ to 1.0 in the streamwise direction. It extends from the wall to a distance $y/L = 1.0$ in the freestream.

Figure 1 also shows a typical mesh obtained after several adaptive cycles. Note that element size increases from left to right as the boundary layer thickens. This test case retains many nonlinear terms in the Navier-Stokes equations thus providing a good test case for comparison. Computations were performed using the exact error to drive the adaptive process. At each adaptation both error estimators are computed and compared to the exact error. The efficiency index (ratio of the estimated error to the true error) is a good measure of the reliability and accuracy of any error estimator. Figure 2 is a plot of the efficiency index vs the number of cycle of adaptation. It clearly shows that the element-by-element estimator is more accurate when the mesh is adapted.

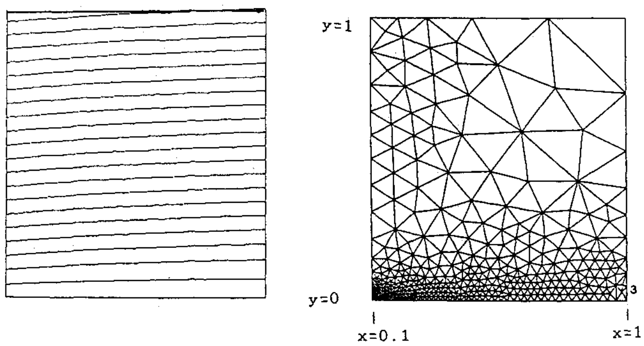

Fig. 1 Two-dimensional boundary layer streamlines and mesh.

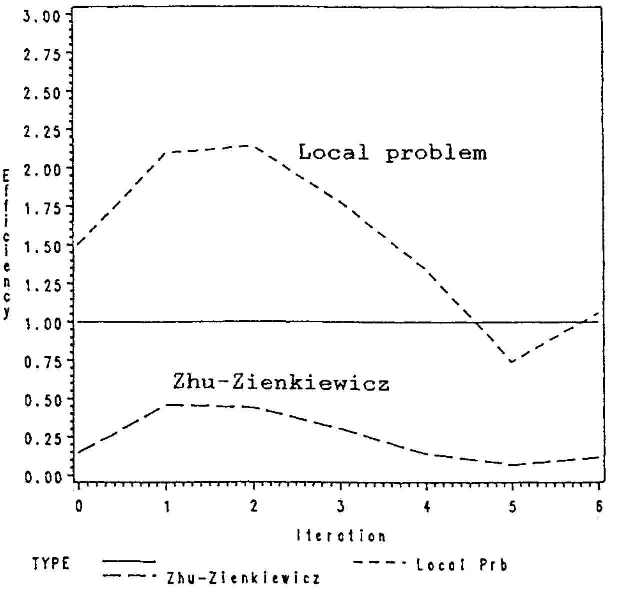

Fig. 2 Efficiency index of error estimator.

In order to test the ability of both estimators to properly guide the adaptation process, the simulations are performed again but this time, the error estimators are used to drive the adaptation. Tables 1 and 2 summarize the results. As can be seen, the element-by-element estimator produces more accurate solutions with fewer degrees of freedom. Furthermore, its efficiency index is closer to one as adaptation proceeds, thus confirming that this estimator provides more accurate estimates of the quality of a given solution.

Flow in an Obstructed Channel

In this section we illustrate the usefulness of the proposed adaptive method by solving a laminar flow of practical interest. The problem consists in solving the flow in an obstructed channel (see Fig. 3). This configuration is characteristic of many practical devices such as labyrinth seals in gas turbines. The channel has unit height and the fence blocks 75% of the channel flow section. The Reynolds number based on the channel height is 82.5. Although this may seem low, the complexity of the flowfield makes this problem a difficult one and an excellent test case.

Figure 3 shows the initial coarse mesh used and provides a sketch of the flowfield. Flow is from left to right. The fence causes significant blockage (75%) in the channel. This results in a complex flow with large streamline curvature, a high-speed jet, and shear layers. A large recirculation zone is present just downstream of the obstacle. The oblique reattaching jet flow causes high shear in the flow and induces a weak separation on the top wall.

Figure 4 shows results obtained after four cycles of adaptation. Notice the very fine grid around the upstream corner of the obstacle. This fine mesh confines the effect of the corner singularity on a small neighborhood of the corner. This figure also illustrates the refinement of the mesh in the shear layer formed by the jet issuing from the gap above the obstacle. The mesh was also graded near the impingement point on the lower wall. Resolution of the oblique stagnation point flow has markedly improved.

The mesh has achieved equidistribution of the error. The error is constant on all elements (3% variation were observed). The finite element solution is thus uniformly accurate throughout the domain. Large elements have been generated in both recirculation zones, since both velocity gradients and pressure do not vary much in these regions.

Increasing the element density in the recirculation zone would result in smoother streamlines. However, since the jet and the recirculation zone are caused by the squeezing of the flow near the fence, meaningful resolution improvements in these regions can only be achieved if resolution is also improved in the region upstream of the obstacle. The equidistribution of the error imbedded in the adaptive strategy is designed to achieve such uniform improvements of the solution.

Figures 5–8 compare experiments with predictions obtained using both error estimators. Predictions were obtained after four cycles of adaptation. At each cycle the target error was set to 20% of the estimated error. As can be seen, both error estimators perform well at the backface of the obstacle ($x/S = 0.0$) and at 1.2 obstacle height ($x/S = 1.2$). However, at stations $x/S = 4.0$ and $x/S = 5.0$, differences appear. For the postprocessing estimator, labeled Zhu, the quality of predictions degrades somewhat at $x/S = 4$ and is poor at $x/S = 5$. At this station, located just downstream of the reattachment point, the flow is very weak, but there are strong pressure gradients. The postprocessing estimator cannot sense this phenomenon and produces a grid that is not sufficiently fine to properly resolve the sharp pressure gradients.

On the other hand, results obtained with the element-by-element estimator are clearly superior at both stations, indicating that, contrary to claims made in Ref. 6, errors due to both the continuity equations and pressure gradients can significantly affect the results.

Computational Efficiency

Results from the preceding section illustrate the accuracy improvements that can be achieved with adaptivity. The proposed adaptive strategy also results in a highly cost-effective solution algorithm that is well worth the added complexity.

Table 3 contains computational statistics obtained on an IBM E/S 9000 with vector facility for the flow over a fence using the element-by-element error estimator. Timings include all aspects of computations (grid generation, flow solution, error estimation, and interpolation of the solution between grids). Computation of the error estimate typically represents

Table 1 Adaptation with postprocessing error estimator

Mesh	No. of equations	Error	Efficiency
0	139	0.28645	0.14
1	251	0.19769	0.30
2	617	0.11076	0.27
3	983	0.03822	0.30
4	2067	0.2602	0.12

Table 2 Adaptation with local problem estimator

Mesh	No. of equations	Error	Efficiency
0	139	0.28645	1.50
1	195	0.23000	2.09
2	387	0.11119	2.14
3	741	0.04719	1.34
4	1495	0.02528	1.05

Table 3 Computational statistics for adaptation

Cycle	0	1	2	3
Grid points	1427	1299	2507	3511
Mesh generation, s	0.89	1.25	1.71	1.84
Solution, s	89.0	40.0	70.0	71.0
Adaptation, s	1.9	2.2	4.05	4.9

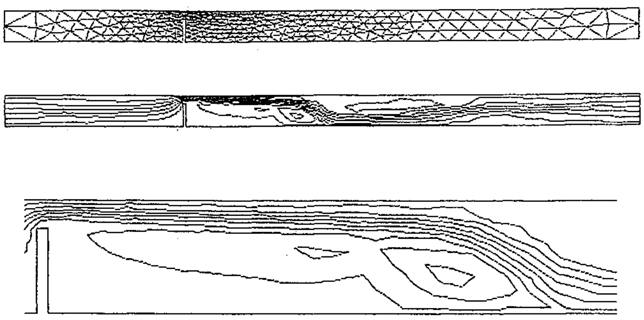

Fig. 3 Initial mesh and flowfield.

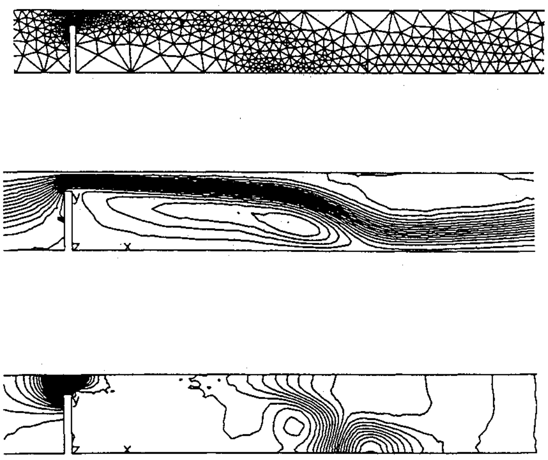

Fig. 4 Results after four adaptive cycles.

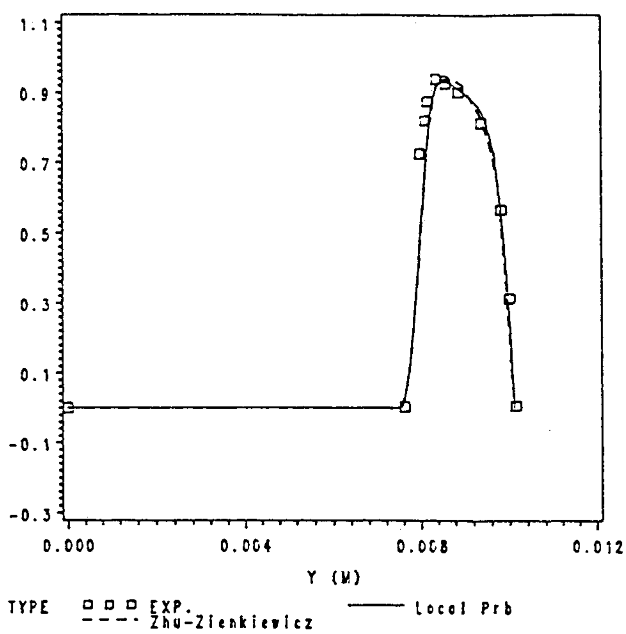

Fig. 5 Axial velocity at $x/S = 0$.

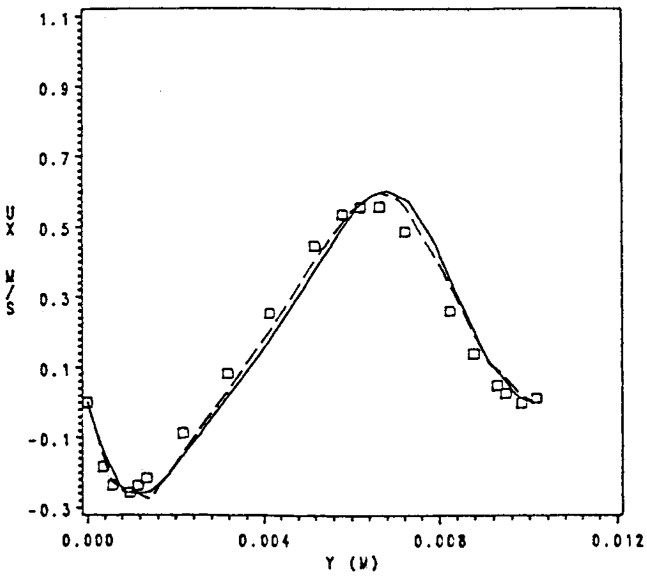

Fig. 6 Axial velocity at $x/S = 1.2$.

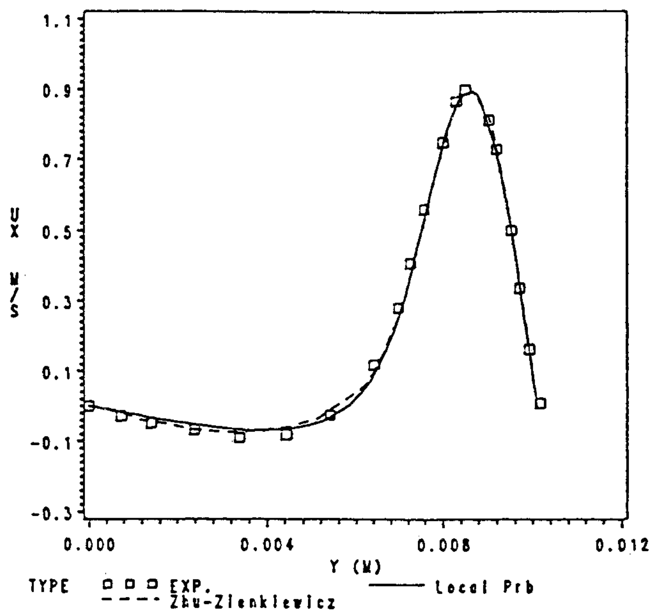

Fig. 7 Axial velocity at $x/S = 4$.

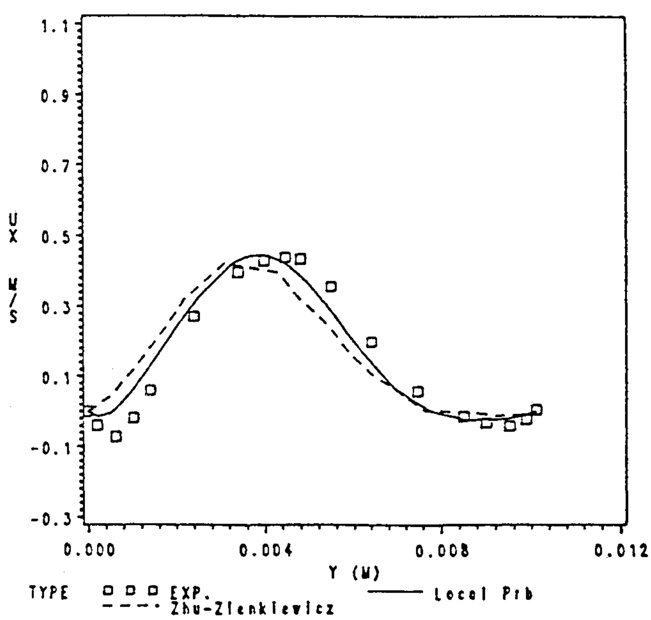

Fig. 8 Axial velocity at $x/S = 5$.

$<8\%$ of the cost of obtaining a solution on a given mesh. The finest mesh requires <20 Mbytes of storage.

Complete solution of this problem required a total of 332 CPU seconds. Solving the same problem directly on the final mesh without using intermediate grids would have required approximately 1300 CPU seconds. The adaptive strategy is thus seen to offer a three- to four-fold reduction in CPU requirement compared to a direct brute force solution of the final mesh. In fact, the advantage offered by adaptivity increases with the fineness of the final mesh. Indeed, the bulk of the computations involving the nonlinear terms is carried out with the coarser meshes and is thus relatively inexpensive. Because good initial guesses are generated on intermediate meshes, only one or two iterations are typically required on the final mesh.

It should also be noted that without adaptivity it would have been nearly impossible to generate a grid leading to comparable accuracy without nearly quadrupling the number of grid points. In fact, it is very difficult to achieve a good allocation of grid points without the extra knowledge gained from the error estimates. Given that Gaussian elimination is used at each Newton iteration, the increase in computational cost is proportional to the cube of the number of grid points. It follows that nonadaptive computations of comparable accuracy would have been approximately 10–20 times more expensive than the adaptive ones.

Conclusion

An adaptive remeshing procedure has been presented for solving complex viscous incompressible flow problems that can be simulated by the Navier-Stokes equations.

Two error estimators have been compared. The first one is based on postprocessing of the stress and strain rate tensors. Although it is sensitive to shear layers and regions of high strain, it does not sense pressure directly. This results in poor solutions in regions where pressure variations dominate the flow.

The second estimator solves a set of partial differential equations for the velocity and pressure errors and is sensitive to both velocity and pressure variations. Solution for the errors proceeds on an element-by-element basis. Results obtained with this estimator are more accurate and the error estimates are more reliable. This approach results in a very cost-effective overall solution strategy.

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